Constrained Kelly Portfolios under α-stable laws

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Motivation: The Kelly Criterion

- Wealth in horizon T, given discrete returns $X_t \in \mathbb{R}^k$

$$W_T(f) = W_0 \prod_{t=0}^{T} \left(1 + \sum_{j=1}^{k} f_j X_{j,t} \right) \tag{1}$$

- *Growth-optimal* fraction $f = [f_1, f_2, \ldots, f_k]^\top$ ▸ Breiman (1961)
- Myopic solution, following Kelly (1956) and Breiman (1961)

$$f^* = \underset{f \in \mathbb{R}^k}{\operatorname{argmax}} \left[\mathsf{E}\left\{ \log W_T(f) \right\} \mid \sum_{j=1}^{k} f_j \leq 1 \right] \tag{2}$$

Motivation: Stable Laws

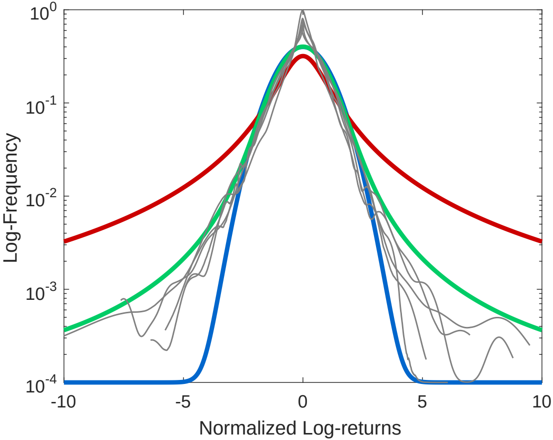
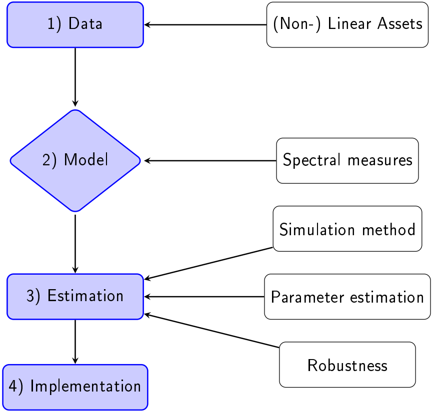


Figure 1: Normalized log-densities of all assets and
Gaussian ($\alpha = 2$), Stable ($\alpha = 1.7$), Cauchy ($\alpha = 1$)

Outline


1) Data
(Non-) Linear Assets
2) Model
Spectral measures
Simulation method
3) Estimation
Parameter estimation
Robustness
4) Implementation

Linear Assets

- $n = 4331$ days of price data (Bloomberg)

Descriptives	DAX30	S&P500	EMUSOV
$\mu \times T$ (in %)	4.74	4.07	3.65
$\sigma \times T^{1/2}$ (in %)	24.88	20.22	1.48
Skewness	0.04	-0.12	0.17
Kurtosis	6.22	10.12	22.33
α	1.69 ± 0.06	1.59 ± 0.06	1.60 ± 0.06
β	-0.20 ± 0.17	-0.16 ± 0.16	0.00 ± 0.14
$\gamma \times T^{1/\alpha}$	0.24	0.22	0.02
$\delta \times T$	0.17	0.16	0.04

Table 1: Log-Return descriptives with ML estimates under α-stability, 1997-2015 including confidence intervals (99%) for stability α and skewness β

Nonlinear Assets

- Long Puts (128) of DAX and S&P 500 as nonlinear functions of the underlyings in T

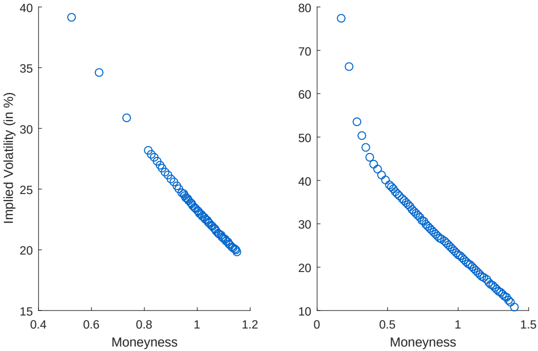


Figure 2: Implied Volatility over Moneyness (Strike/Price)

Spectral Measures for growth

- Spectral Measure

$$M_\phi \left\{ W_T(f_t) \right\} = \int_0^1 \phi(x) F^{-1}_{W_T}(x) dx \tag{3}$$

- Growth Measures $G_\phi \left\{ W_T(f_t) \right\}$
  - Arithmetic mean for

$$\phi_{\mathsf{E}}(x) = 1 \tag{4}$$

  - Geometric mean for ▸ Special case under Gaussianity

$$\phi_{\mathsf{Elog}}(x) = \log \tag{5}$$

  - Median for

$$\phi_{\mathsf{Median}}(x) = \delta(x = 0.5) \tag{6}$$

Spectral Measures for security

- Unrestricted Kelly optimization - larger bets than a risk-averse investor would accept (Hausch and Ziemba, 1985)
- Security Measures $S_\phi\left\{W_T(f_t)\right\}$
  - Quantile for

$$\phi_{\mathsf{Q}_\alpha}(x) = \delta(x = \alpha),\ 0 \le \alpha \le 1 \tag{7}$$

  - Expected Shortfall for

$$\phi_{\mathsf{ES}_\alpha}(x) = \alpha^{-1} 1(x < \alpha) \tag{8}$$

Growth-Security Frontier

- Set of possible combinations

$$U = [G\{W_T(f)\}, S\{W_T(f)\}],$$

given f feasible

- Efficient points given as

$$U^* = [G\{W_T(f^*)\}, S\{W_T(f^*)\}]$$
$$f^* = \underset{f \in \mathbb{R}^k}{\operatorname{argmax}} [G\{W_T(f)\}]$$
$$\text{s.t. } S\{W_T(f)\} \leq b, \;\; b \in \mathbb{R}$$

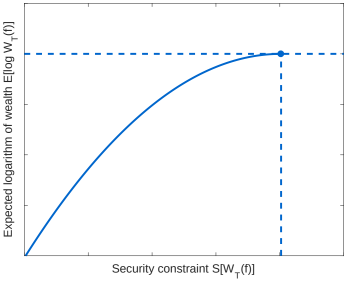


Figure 3: Growth-security frontier

The class of Lévy-Stable Distributions

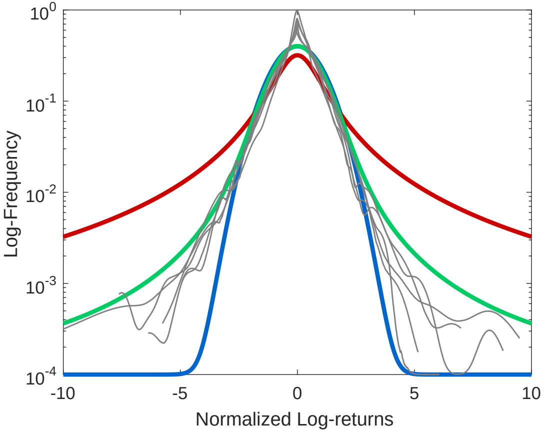


Figure 4: Normalized log-densities of all assets and
Gaussian ($\alpha = 2$), Stable ($\alpha = 1.7$), Cauchy ($\alpha = 1$)

Elliptically contoured stable distributions

- Let $W \sim S(\alpha/2, \beta, 1, 0)$, $0 < \alpha < 2$, $\beta = (\cos \pi\alpha/4)^{2/\alpha}$ and $G \sim \mathrm{N}(0, \Gamma)$, $\Gamma = AA^\top$ ▸ Univariate Stable
- Then X is elliptically contoured stable if

$$\begin{aligned} X &\overset{\mathcal{L}}{=} W^{1/2}G + \delta \\ &\overset{\mathcal{L}}{=} W^{1/2}AZ + \delta,\ Z \sim \mathrm{N}(0, I_k) \\ &\overset{\text{def}}{=} AY + \delta,\ Y \sim E_k(\alpha, 0, I_k, 0) \end{aligned} \tag{9}$$

- The according characteristic function of X is

$$\varphi_X(u) = \mathrm{E}\left(iu^\top X\right) = \exp\left\{-\left(\frac{1}{2}u^\top \Gamma u\right)^{\alpha/2} + iu^\top \delta\right\}, \tag{10}$$

Γ being the scale and $\delta \in \mathbb{R}^k$ location

Nonparametric Scaling Approximation

- Elliptically contoured stable distribution

$$X_t \sim E_k(\alpha, 0, \Gamma_t, \delta_t),\ \Gamma_t = A_t A_t^\top \tag{11}$$

- Distribution for frequency $T,\ t \mid T$

$$X_T = T \times X_t \sim E_k(\alpha, 0,\ T\Gamma_t,\ T\delta_t) \tag{12}$$

- Scale the non-parametric daily distribution of X_t from daily frequency t to frequency T
  - Normalization to radially symmetric stable

$$Y = (X_t - \delta_t)\ A_t^{-1},\ \ Y \sim E_k(\alpha, 0, I_k, 0) \tag{13}$$

  - Scale to horizon distribution in T

$$X_T = \delta_T + AY,\ X_T \sim E_k(\alpha, 0, \Gamma_T, \delta_T),\ \Gamma_T = AA^\top \tag{14}$$

Nonparametric Scaling Approximation

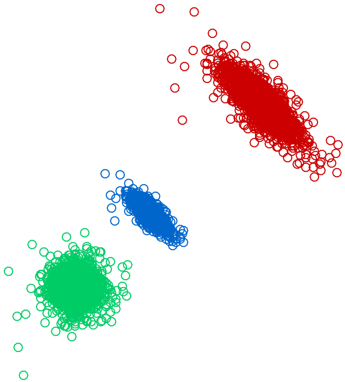


Figure 5: Changes in location and scale (two dimensions):
$X_t \sim E_2(\alpha, 0, \Gamma_t, \delta_t)$, $Y \sim E_2(\alpha, 0, I, 0)$ and $X_T \sim E_2(\alpha, 0, \Gamma_T, \delta_T)$

Parameter Estimation

- Fractional Moments

$$E|X|^p = \int_{-\infty}^{+\infty} |X|^p f(X) dX \tag{15}$$

  - finite for $0 < p < \alpha$
  - infinite for $p \geq \alpha$
- Under simulation
  - stability α, scale Γ and location δ need to be estimated
- Under scaling approximation
  - stability α not necessary

Estimating the location δ_t

- For elliptical stable laws with $1 < \alpha \leq 2$

$$\delta_t = \mathsf{E}\, X_t < \infty \tag{16}$$

- Decision theory
  - 0-1 loss (Nolan, 1997)
  - quadratic loss
- Lack of theory for shrinking under α-stability as in Hansen (2015)

Estimating the scaling matrix Γ_t

- Given that X is elliptically stable,

$$\forall u, \;\; u^\top X \sim S(\alpha, 0, (u^\top \Gamma u)^{\frac{1}{2}}, u^\top \delta) \tag{17}$$

- The $k(k+1)/2$ parameters of the scale matrix Γ are estimated by

$$\Gamma_{j,j} = \gamma_j^2 \tag{18}$$

$$\Gamma_{j,i} = \frac{1}{2}\left\{\gamma^2(1,1) - \gamma_{i,i}^2 - \gamma_{j,j}^2\right\}, \tag{19}$$

- $\gamma^2(1,1) = (1,1)^\top(X_j, X_i) = X_j + X_i$ and γ_j is the univariate scale parameter of asset j, estimated by MLE
- $\Gamma_{j,i}$ depends solely on directions $(1,1)$,$(1,0)$ and $(0,1)$

Optimization

- Specific optimization with
  - 5 linear assets (long only)
  - 1 short asset (short only)
  - 128 non-linear assets (long only)
- Quantile restricted optimization
- $W_0 = 100,\ 0 \leq b \leq 1$ with $\alpha = 0.001$

$$
\begin{aligned}
f^* = \underset{f \in \mathbb{R}^k}{\operatorname{argmax}}\ & G_{\phi_{\mathbf{Elog}}} \{W_T(f)\} \\
\text{s.t. } & S_{\phi_{\mathbf{Q}_\alpha}} \left\{ 1 - \frac{W_T(f)}{W_0} \right\} \leq b, \\
& \sum_{j=1}^{134} f_j \leq 1
\end{aligned}
\tag{20}
$$

Discrete wealth return distribution, $b = 0.15$

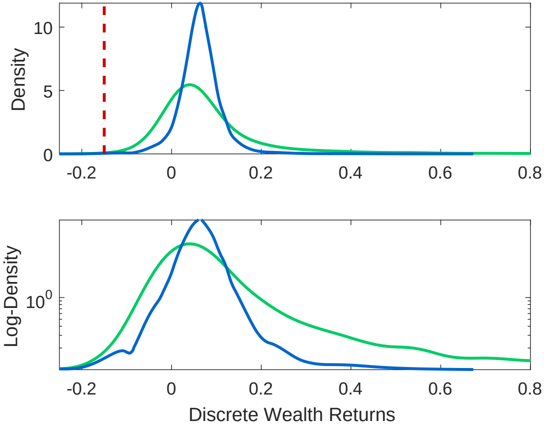


Figure 6: Discrete wealth returns without and with Options

Discrete wealth return statistics, $\alpha = 0.001$

Portfolio (in %)	Without	With
Geometric mean	6.03	8.85
Arithmetic mean	6.33	10.9
Standard deviation	4.98	26.94
Minimum	−29.76	−28.19
$Q_{0.1\%}$	−15.0	−15.0
$Q_{1\%}$	−6.02	−8.1
$Q_{10\%}$	1.33	−0.95
$Q_{50\%}$	6.25	5.27
$Q_{90\%}$	11.3	23.35
$Q_{99\%}$	20.92	119.12
Maximum	67.26	871.57

Table 2: : Discrete wealth return statistics (in %) without and with options

Fractions	Without	With
DAX30	0.01	0.74
S&P500	0	1.29
EMUSOV	2.36	1.88
EMBI	0.33	0.48
BCI	0	0
SHORT	−1.69	−3.89
Put DAX	/	0.14
Put S&P500	/	0.36

Table 3: : Portfolio fractions without and with options

Improvement in terms of quantile, $\alpha = 0.001$

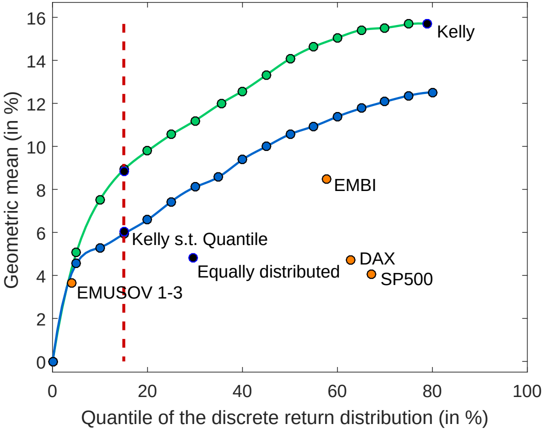


Figure 7: Kelly-Quantile-Frontier

without and with Options

Conclusion

- Constrained Kelly optimization - growth-optimal strategy, given personal preferences
- Non-linear assets - beneficial for quantile and Expected Shortfall restrictions
- Stable laws - non-normal limiting behavior for financial market returns

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General i.i.d. - Breiman (1961)

▸ Motivation

- Investment strategy $\Lambda = \begin{bmatrix} f_{t,j} & \cdots & f_{T,j} \\ \vdots & \ddots & \vdots \\ f_{t,k} & \cdots & f_{T,k} \end{bmatrix} = [f_t \cdots f_T]$
  - investment fractions f_t from time t to $T \in \mathbb{N}^+$
  - opportunities j to $k \in \mathbb{N}^+$
- Security price vector $p_t = \begin{bmatrix} p_{t,j} \\ \vdots \\ p_{t,k} \end{bmatrix}$
- Return per unit invested $x_t = \begin{bmatrix} \frac{p_{t,j}}{p_{t-1,j}} \\ \vdots \\ \frac{p_{t,k}}{p_{t-1,k}} \end{bmatrix}$.

Asymptotic outperformance

Theorem

- *Myopic log-optimal strategy* $\Lambda^* = [f^* \ \cdots \ f^*]$
- *Significantly different strategy* Λ

$$\mathsf{E}\{\log W_T(\Lambda^*)\} - \mathsf{E}\{\log W_T(\Lambda)\} \longrightarrow \infty, \tag{21}$$

- *Kelly investor dominates asymptotically*

$$\lim_{T\to\infty} \frac{W_T(\Lambda^*)}{W_T(\Lambda)} \xrightarrow{a.s.} \infty \tag{22}$$

Leo Breiman on BBI: 

Minimize time to reach goal g

Theorem

- *Let $T(g)$ be the smallest T, such that $W_t \geq g, \; g > 0$*
- *If equation (21) holds,*

$$\exists \alpha \geq 0 \perp\!\!\!\perp \Lambda, \; g \tag{23}$$

such that

$$\mathrm{E}\{T^*(g)\} - \mathrm{E}\{T(g)\} \leq \alpha, \tag{24}$$

- $\perp\!\!\!\perp$ *- independent of*
- Λ^* *asymptotically minimizes the time to reach goal g*

Time invariance

Theorem

- *Given a fixed set of opportunities the strategy is*
  - *fixed fraction*
  - *independent of the number of trials T*

$$\Lambda^* = [f_1^* \cdots f_T^*], \;\; f_1^* = \cdots = f_T^* \tag{25}$$

The class of Lévy-Stable Distributions

▸ Elliptically contoured stable distributions

- Fourier transform of characteristic function $\varphi_X(u)$

$$S(X \mid \alpha, \beta, \gamma, \delta) = \frac{1}{2\pi} \int \varphi_X(u) - \exp(-iux) du$$

- Characteristic function representation, $0 < \alpha < 2, \alpha \neq 1$

$$\log \varphi_X(u) = iu\delta - \gamma |u|^{\alpha} \{1 + i\beta (u/|u|) \tan (\alpha\pi/2)\} \quad (26)$$

- Stability or invariance under addition

$$n \log \varphi_X(u) = iu(n\delta) - (n\gamma)|u|^{\alpha} \{1 + i\beta (u/|u|) \tan (\alpha\pi/2)\}$$

- Limiting distribution of n i.i.d. stable r.v., $0 < \alpha \leq 2$

$$n^{-\frac{1}{\alpha}} \sum_{i=1}^{n} (X_i - \delta) \xrightarrow{\mathcal{L}} S(\alpha, \beta, \gamma, 0) \quad (27)$$

Unconstrained Kelly fraction for $\alpha = 2, \ \beta = 0$

▸ Spectral measures for growth

- $X \sim \mathrm{N}(\mu, \Sigma)$ and risk free rate $r > 0$

$$W_n(f) = W_0 \left\{ 1 + r + f^\top (X - r) \right\} \tag{28}$$

- Taking logarithm and expectations on both sides leads to $\mathrm{E}\left[\log\left\{W_n(f)/W_0\right\}\right]$, which is expanded in a Taylor series

$$g(f) = \mathrm{E}\left\{ \log(1+r) + \frac{1}{1+r}(\mu - 1r)^\top f - \frac{1}{2(1+r)^2} f^\top \Sigma f \right\} \tag{29}$$

- From quadratic optimization (Härdle and Simar, 2015)

$$f^* = \Sigma^{-1}(\mu - 1r) \tag{30}$$

$$g_\infty(f^*) = r + f^{*\top} \Sigma f^*/2 \tag{31}$$

Unconstrained Kelly fraction for $\alpha = 2,\ \beta = 0$ $\mu = [0.03\ 0.08],\ \sigma = [0.15\ 0.15],\ \rho = 0,\ r = 0.01$

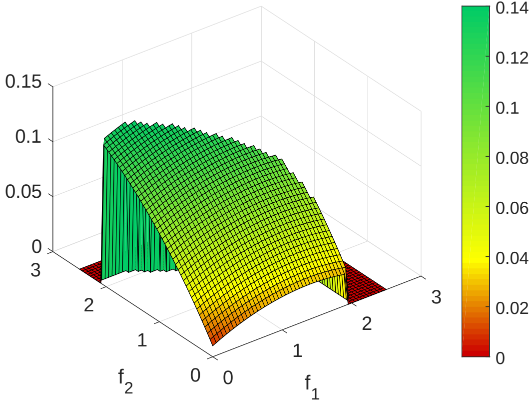

Figure 8: Exponential growth rate g(f) (Gaussian)

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